Periodic orbit quantization of Baker's map

1994 J. Phys. A: Math. Gen. 27 4449

(http://iopscience.iop.org/0305-4470/27/13/019)

J. Phys. A: Math. Gen. 27 (1994) 4449–4455. Printed in the UK

Periodic orbit quantization of Baker's map

Bruno Eckhardt† and Fritz Haake‡

† Fachbereich Physik und ICBM der C.v. Ossietzky Universität, Postfach 25 03, D-26111 Oldenburg, Germany

‡ Fachbereich Physik, Universität GH Essen, D-45117 Essen, Germany

Received 31 January 1994

Abstract. We consider the resummation of the semiclassical Selberg zeta function for quantized maps on compact phase space, specifically the quantized Baker's map. In particular, we demonstrate that the semiclassical periodic-orbit expansion leads to an effectively finite polynominal for the spectrum of the quantum map. However, the coefficients are not self-inverse (as required by unitarity). An extension of the semiclassical approximation by including corrections due to the discontinuities of the classical map improves the situation. The improved polynominal is closer to being self-inverse but the eigenphases remain complex. Slightly better results are obtained if the functional equation is imposed on the semiclassical expansion.

1. Introduction

Several lines of thought have been followed in the quest for semiclassical quantization of chaotic systems. The Gutzwiller trace formula [1, 2] combined with the cycle expansion [3, 4] to eliminate convergence problems has met with some success [5–8]. By analogy with the Riemann zeta function [9, 10] and the Selberg zeta function from the theory of geodesic motion on surfaces of constant negative curvature [11, 12], the unitarity of quantum evolution [13] and the general properties of the spectral determinant [14, 15] suggest a functional equation which could be put to good use [16–18]. Studies of quantizations of Poincaré maps [13, 19] and scattering systems [20] have been very elucidating theoretically: they suggest that convergence and the functional relation arise from the fact that in a bounded phase space Planck's constant introduces a natural smallest volume and the phase space should be divided into a finite number of cells of this volume. Bogomolny [13, 19] achieves this reduction through a semiclassical approximation and Doron and Smilansky [20] through the omission of closed scattering channels. In none of the previous studies in bounded systems could these properties be unambigously demonstrated for the semiclassial trace formula.

To be able to study the behaviour of semiclassical expansions most clearly and independently from approximations on the quantum side, we turn here to a study of quantized maps on compact phase spaces. Then the quantization procedure automatically demands a finite unitary matrix for the time evolution. Specifically, we study Baker's map since its classical mechanics is well understood and can be mapped onto a complete shift on two symbols so that the prerequisites for cycle expansion are also satisfied. We can draw here on the extensive studies of the quantum and semiclassical mechanics of Baker's map by Balazs, Saraceno and Voros [21–24]; for the quantum map we use the symmetry-preserving quantization of Saraceno [22].

In the next section we present a summary of previous work, and our results using the primitive and an improved semiclassical expansion. We close with a discussion in section 3.

0305-4470/94/134449+07$19.50

2. Baker's map

2.1. Exact results

We briefly recapitulate the main features of Baker's map as needed here. The classical map acts on the unit square $[0,1]\times[0,1]$ and maps points (q,p) according to

$$\begin{aligned} q' &= 2q \qquad (\mathrm{mod}\ 1) \\ p' &= (p+[2q])/2 \end{aligned} \tag{1}$$

where $[x]$ denotes the largest integer $\leqslant x$. There exists a one-to-one mapping of all trajectories onto symbol sequences of 0's and 1's, depending on the itineraries of a point lying to the left (when the symbol 0 is assigned) or to the right (the symbol then being 1) of the line $x=\frac{1}{2}$, the line of discontinuity. Periodic orbits are given by periodic symbol strings. They are all unstable, with stability exponent $n\ln 2$, if n is the period (i.e. the largest eigenvalue of the linearization around the orbit is 2^n). For the semiclassical calculation one also needs the action associated with a trajectory [23]

$$S_\nu = \frac{\nu\overline{\nu}}{2^n-1} \qquad (\mathrm{mod}\ 1) \tag{2}$$

for a cycle of period n. The integers ν and $\overline{\nu}$ are obtained from the symbol string $\{i_1,\dots,i_n\}$ when considered as a binary representation of a number

$$\nu=\sum_{k=1}^{n} i_k 2^{k-1} \qquad \overline{\nu}=\sum_{k=1}^{n} i_k 2^{n-k}. \tag{3}$$

Since the dimension of the Hilbert space and thus the inverse of Planck's constant is integer, one can take S modulo 1. This also ensures that points for which the symbolic codings differ by only one cyclic permutation have the same actions.

Since the quantized map is represented by a finite unitary matrix U of dimension N, eigenvalues are given by the inverses of zeros of the polynominal

$$Z_N(z)=\det(1-zU)=\sum_{k=0}^{N} a_k z^k. \tag{4}$$

(Of course, for actual numerical calculations one would diagonalize the matrix and not search for zeros of the polynominal. We use it here only to stress the similarity to the semiclassical approach.) Because of unitarity, the coefficients satisfy [13, 19] $a_k^* = a_{N-k}\mathrm{e}^{\mathrm{i}\theta}$ with $\mathrm{e}^{\mathrm{i}\theta}=(-1)^N\det U$, a property commonly known as self-inversiveness.

2.2. Semiclassical approximation

A semiclassical expression for $Z(z)$ (distinguished from the exact one by the index 'sc') may be obtained by noting the semiclassical trace formula

$$(\mathrm{tr}\, U^n)_{\mathrm{sc}} = \sum_{P\in\mathrm{Fix}(n)} A_P \mathrm{e}^{\mathrm{i}S_P/\hbar} \tag{5}$$

in which the sum extends over all points P periodic after n iterations with action S_P and weight $A_P = 2^{n/2}/(2^n - 1)$ and the relation $\det M = \exp(\operatorname{tr} \ln M)$, whereby

$$Z_N^{(\mathrm{sc})} = \exp\left(- \sum_{k=0}^{\infty} \frac{z^k}{k} (\operatorname{tr} U^k)_{\mathrm{sc}} \right) = \sum_{k=0}^{\infty} a_k^{(\mathrm{sc})}(N) z^k. \tag{6}$$

Planck's constant can take on values $\hbar = 1/2\pi N$, where N is the dimension of the unitary evolution operator. Since the quantum map takes on a particularly simple form for even N, only such values will be considered here.

Note that there is no *a priori* reason for the semiclassical expression (6) to end at the finite order N as does the exact quantum determinant (4). When expanded, the coefficients $a_k^{(\mathrm{sc})}$ contain complicated groupings of all traces $\operatorname{tr} U^n$ for $1 \leqslant n \leqslant k$ [25–27]. For these coefficients to vanish, it would be necessary for there to be strong correlations between the long and short trajectories—and indeed it has been argued that these exist in connection with the cycle expansion [5, 7]. Thus, a first test of the validity of semiclassical theory will be the study of the behaviour of the coefficients as a function of N, i.e. Planck's constant.

Figure 1 shows the absolute values of the coefficients $|a_n^{(\mathrm{sc})}|$ as a function of n for different values of the dimension of the phase space N. One notes fluctuations around some value of order one [28], followed by a rapid drop. Thus the polynominal effectively has a finite degree. As an estimate of the degree of the polynominal, one can take the index beyond which the coefficients stay below 1. This critical index seems to be $N + 3$ rather than N as expected from the quantum propagator which has dimension N.

The case $N = 0$ (corresponding to $\hbar = \infty$) is unphysical but useful to check the calculations. Then the actions drop out of the trace formula (5) and the sums over the eigenvalues can be evaluated using the Euler product formula

$$Z_0^{(\mathrm{sc})}(z) = \prod_{j=0}^{\infty} \left(1 - \sqrt{2}\, 2^{-j} z^j\right) \tag{7}$$

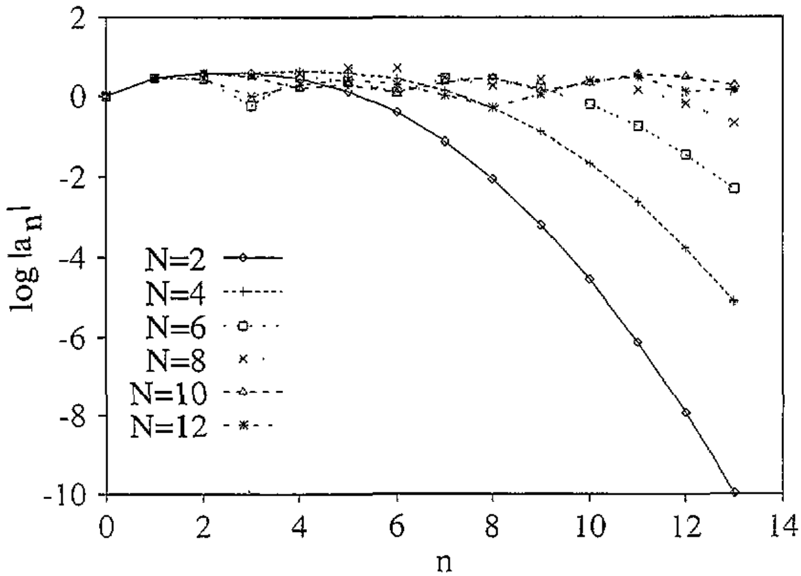

Figure 1. Absolute values of the coefficients of the semiclassical characteristic polynominal (5) plotted against n for different dimension N of the quantum Hilbert space on a semilogarithmic scale.

$$= \sum_{k=0}^{\infty} z^k \left(\sqrt{2}/2\right)^k \frac{2^{-k(k+1)/2}}{\prod_{j=1}^{k}(1-2^{-j})} \tag{8}$$

clearly showing the faster than exponential decay of the coefficients with k for sufficiently large values; note, however, that the coefficients first rise before submitting to the decay. As figure 1 shows, the decline of the coefficients is faster than exponential even for non-zero N.

The next quantity one might like to study is the self-inversiveness [29]. But since the relevant coefficients are not uniquely singled out (semiclassically, that is), this makes little sense. The eigenvalues computed from the semiclassical polynominal do converge with increasing n but are scattered throughout the plane without any immediate relationship with the exact eigenvalues.

2.3. Improved semiclassical approximation

In fact, as noted by Voros and Saraceno [24], the stationary phase approximation, which is the central semiclassical approximation, fails rather badly for Baker's map, presumably because of the discontinuities inherent in the map. Taking this partially into account [30], one can derive a correction factor to the weights of periodic orbits in (5), i.e.

$$(\mathrm{tr}\, U^n)_{\mathrm{sq}} = \sum_{P\in \mathrm{Fix}(n)} A_P \mathrm{e}^{2\pi \mathrm{i} S_P N} R_P(n, N) \tag{9}$$

with $R = 1$ unless 2^n divides N, in which case

$$R_P(n, N) = \frac{2^n - 1}{N} \sum_{n,m=1}^{N} \exp(-2\pi \mathrm{i}(2^n - 1)(n - \tfrac{1}{2} - Mq)(m - \tfrac{1}{2} - Mp)/N) \tag{10}$$

where (q, p) are the position and momentum coordinates of the point P and $M = N/2^n$. This correction mainly affects points near the boundary along the lines of discontinuity. Note that this 'semiquantum' approximation in many cases yields better results than the semiclassical one.

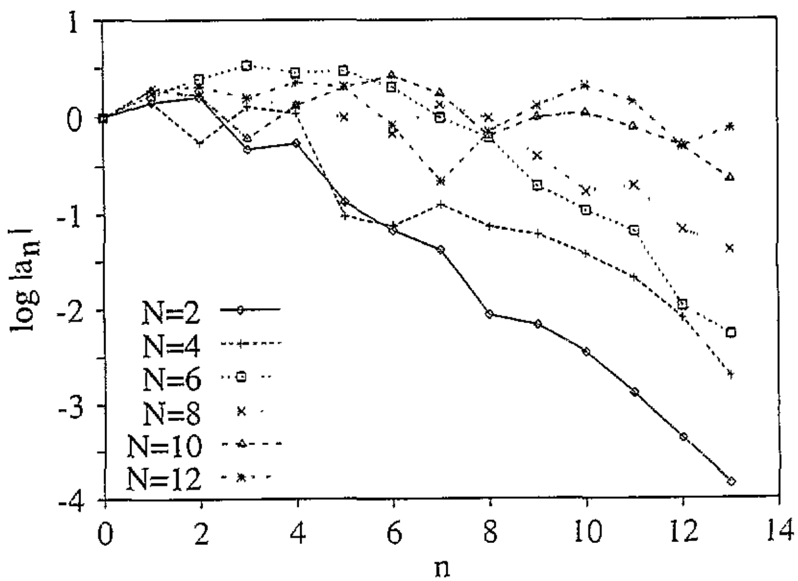

Figure 2. As figure 1, but for the coefficients of the semiquantum polynominal.

Because of the requirement that the corrections apply only if 2^n divides N, for a given N only a few traces are effected. Thus, one can write

$$Z^{(sq)}(z) = Z^{(sc)}(z)\mathrm{e}^{P_M(z)} = \sum_n a_n^{(sq)} z^n \tag{11}$$

where $P_M(z)$ is a polynominal in z of degree $M = N/2^n$. For the calculations, the whole expression is again expanded in a power series in z and truncated at the highest power n for which all periodic orbits are available. Thus this correction does not change the asymptotic decay of the coefficients. Actually, as shown in figure 2, it acts so as to cause this decay to set in earlier, roughly near $n = N$, as required by the quantum map.

Table 1. Coefficients of the semiclassical, semiquantum and exact polynominal for $N = 6$. The first coefficient for $n = 0$ always equals 1. The exact expansion terminates at $n = N = 6$.

n	$a_n^{(sc)}$	$a_n^{(sq)}$	$a_n^{(exact)}$
1	$2.8284 + i0.0000$	$-0.8214 + i1.4880$	$-0.8214 + i1.4880$
2	$2.6667 + i0.0000$	$-2.1031 - i1.2222$	$-1.5396 - i1.2222$
3	$0.4587 - i0.3506$	$2.3709 - i2.3819$	$1.7778 - i1.7778$
4	$-3.2910 + i0.3278$	$1.6943 + i2.2250$	$1.2222 + i1.5396$
5	$2.3546 + i0.5898$	$-2.6007 + i1.3373$	$-1.4880 + i0.8214$
6	$0.8769 - i0.8832$	$-0.3495 - i1.9930$	$0.0000 - i1.0000$
7	$-2.7923 - i0.0456$	$0.9781 - i0.0176$	
8	$2.3253 + i1.2160$	$-0.1207 + i0.6126$	
9	$-0.7140 - i1.3858$	$-0.1928 + i0.0212$	
10	$-0.2144 + i0.5936$	$0.0665 - i0.0865$	
11	$0.1754 - i0.0304$	$0.0644 - i0.0156$	
12	$-0.0151 - i0.0318$	$0.0107 - i0.0007$	

Table 2. Eigenvalues of the quantized map for $N = 6$ as computed from various approximations. All eigenvalues are given in a polar representation $r\mathrm{e}^{\varphi}$; for the exact ones in the last column only the angle is shown since $r = 1$. The first two columns give the zeros as computed from the semiclassical expression (5) and the semiquantum expression (9), respectively. All periodic orbits up to period 13 were used. The resulting polynominal of degree 13 has seven additional roots, which depend sensitively on the truncation; they all lie inside the unit circle. The three closest to the unit circle are listed as well. In the next to last column the semiclassical data for $\mathrm{tr}\,U$, $\mathrm{tr}\,U^2$ and $\mathrm{tr}\,U^3$ were combined with the exact determinant $\det U = -\mathrm{i}$ and a functional equation to arrive at a self-inversive polynominal of degree 6.

Semiclassical		Semiquantum		Functional equation		Exact
r	φ	r	φ	r	φ	φ
1.002	2.858	1.003	2.857	1.003	3.114	2.836
0.915	0.492	0.918	0.491	0.901	0.235	0.461
1.021	0.002	1.019	−0.001	1.112	0.222	0.101
1.121	−1.008	1.123	−1.010	0.996	−0.837	−1.001
0.910	−1.316	0.913	−1.329	1.003	−1.407	−1.367
1.031	−2.735	1.030	−2.733	0.997	−2.898	−2.602
0.798	−0.013	0.829	−0.024			
0.650	0.545	0.611	0.776			
0.616	0.929	0.539	−2.559			

The coefficients obtained from this expansion are slightly closer to being self-inverse, but again it is not satisfied exactly. In table 1 we list the coefficients for $N = 6$, where the corrections from equation (9) have been applied to the fixed points at $n = 1$ only. In table 2 we list the eigenvalues as found by four different methods: first from the primitive semiclassical approximation (5), second from the semiquantum approximation (9), third from a combination of the semiquantum approximation for the first $N/2$ coefficients with the exact determinant through the functional equation and, finally, from an exact diagonalization of the matrix. In the first two cases, one also has additional zeros since the higher-order coefficients do not vanish. The data show that the best results are achieved for a combination of the semiclassical traces with the functional equation.

3. Conclusions

The calculations for Baker's map have unambiguously shown that long orbits are correlated with short orbits in such a way that the coefficients of the determinant (5) decay for n larger than N, the inverse of Planck's constant [5, 7]. This decay is faster than exponential and thus faster than for two-degree-of-freedom systems where the asymptotic behaviour of the semiclassical expression is dominated by a pole [8].

The point after which this decay sets in is larger than N in the semiclassical case, but this can be improved by including corrections to the stationary phase approximation. Then the coefficients are self-inverse and the eigenvalues approach the unit circle. There remains a rather large discrepancy between the effort that one has to put into the semiclassical calculation and the accuracy of the results one obtains. It is clear that N essentially independent quantum phases require N numbers as input, which, together with the functional equation, means that all (complex) traces up to $N/2$ have to be determined. Thus roughly $2^{N/2+1}/N$ orbits have to be computed in this example.

One objection one may raise to the present calculation is that Planck's constant has not been small enough, that one has not reached the semiclassical limit proper. However, one might hope that this would affect the accuracy of the results only and not the qualitative properties of the expansion such as convergence, self-inversiveness and the confinement of the eigenphases to the unit circle. Calculations by F M Dittes (private communication) using Bogomolny's transfer operator approach [13] suggest that these properties emerge asymptotically as $N \to \infty$. This might then justify imposing the functional equation [10, 13] or self-inversiveness on the coefficients. This reduces the maximal period required to $N/2$ and also improves the estimates for the eigenvalues.

Acknowledgments

We thank F M Dittes, M Saraceno and A Voros for useful discussions and for sharing their results before publication. BE would like to thank the Institute of Theoretical Physics at the Technion, Haifa, for support during the final stages of this work.

References

[1] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)

[2] Gutzwiller M C 1967 *J. Math. Pure. Appl.* **8** 1979; 1969 *J. Math. Pure. Appl.* **10** 1004; 1970 *J. Math. Pure. Appl.* **11** 1791

[3] Artuso R, Aurell E and Cvitanović P 1990 *Nonlinearity* **3** 325
[4] Artuso R, Aurell E and Cvitanović P 1990 *Nonlinearity* **3** 361
[5] Cvitanović P and Eckhardt B 1989 *Phys. Rev. Lett.* **63** 823
[6] Eckhardt B 1993 *Proc. Int. School of Physics 'Enrico Fermi' (Course CXIX 'Quantum Chaos')* ed G Casati *et al* (Amsterdam: North-Holland) p 77
[7] Eckhardt B, Russberg G, Cvitanović P, Rosenqvist P E and Scherer P 1994 *Quantum Chaos* ed G Casati and B V Chirikov (Cambridge: Cambridge University Press) to appear
[8] Eckhardt B and Russberg G 1993 *Phys. Rep.* E **47** 1578
[9] Berry M V 1986 *Quantum Chaos and Statistical Physics (Lecture Notes in Physics 263)* ed T H Seligman and H Nishioka (Berlin: Springer) p 1
[10] Berry M V and Keating J 1990 *J. Phys. A: Math. Gen.* **23** 4839
[11] Selberg A 1956 *J. Ind. Math. Soc.* **20** 47
[12] Balazs N and Voros A 1986 *Phys. Rep.* **143** 109
[13] Bogomolny E B 1990 *Comm. At. Mol. Phys.* **25** 67
[14] Berry M V and Keating J P 1992 *Proc. R. Soc.* **437** 151
[15] Keating J P 1992 *Proc. R. Soc.* **436** 99
[16] Tanner G, Scherer B, Bogomolny E B, Eckhardt B and Wintgen D 1991 *Phys. Rev. Lett.* **67** 2410
[17] Sieber M and Steiner F 1991 *Phys. Rev. Lett.* **67** 1941
[18] Smilansky U 1993 *Nucl. Phys.* A **560** 57
[19] Bogomolny E B 1992 *Nonlinearity* **5** 804
[20] Doron E and Smilansky U 1992 *Nonlinearity* **5** 1055
[21] Balazs N L and Voros A 1988 *Ann. Phys.* **190** 1
[22] Saraceno M 1990 *Ann. Phys.* **199** 37
[23] Ozorio de Almeida A M and Saraceno M 1991 *Ann. Phys.* **210** 1
[24] Saraceno M and Voros A 1992 *Chaos* **2** 99
[25] Plemelj J 1909 *Monat. Math. Phys.* **15** 93
[26] Smithies F 1941 *Duke Math. J.* **8** 107
[27] Reed M and Simon B 1978 *Analysis of Operators (Methods of Modern Mathematical Physics 4)* (New York: Academic)
[28] Kus M, Haake F and Eckhardt B 1993 *Z. Phys.* B **92** 221
[29] Bogomolny E B, Bohigas O and Leboeuf P 1992 *Phys. Rev. Lett.* **68** 2726
[30] Saraceno M and Voros A 1993 Towards a semiclassical theory of the quantum Baker's map *Preprint*